3 A SET OF UNIFORM VARIATIONAL PARAMETERS
FOR SPACE TRAJECTORY ANALYSIS 6

6 S. Pines
H. Wolf
A. Bailie 9

29b Report No. 67-5 END
Contract NAS 5-9085 29A
25

9 February 1967 10

Analytical Mechanics Associates, Inc.
Westbury, L.I., N.Y. 3

TABLE OF CONTENTS

I INTRODUCTION

A set of orbit parameters (α_1, α_2, α_3, α_4, α_5, α_6) is presented which may be used in differential correction schemes. They are being used in the Goddard Minimum Variance Orbit Determination Program.

This set of parameters contains the energy (more precisely $\frac{1}{a}$), all others being independent of the energy. Consequently, the set of parameters, does not exhibit the deterioration in the conditioning of the associated matrices, such as shown if the initial state vector, for instance, is used.

In addition the set is almost completely general allowing for continuous transition from circular to elliptic, and from elliptic to parabolic and hyperbolic orbits. No special measures need be taken for equatorial or near equatorial orbit planes. The set is not, however, suitable for characterizing rectilinear orbits.

The positioning of the orbit plane is achieved by means of rigid rotations of this plane about the velocity and position vectors of the nominal orbit. The orbit in its plane is positioned by a rigid rotation about the nominal angular momentum vector. Thus on the nominal orbit the α_1, α_2, and α_3 are identically zero. The remaining α's are quantities associated with the size and shape of the orbit. For many problems it is convenient to treat the gravitational constant, μ, as an additional variable. This report contains the analytic partial derivatives for the seventh parameter. In addition, the differential state transformation matrix is augmented to a full seven by seven to accommodate this new variable.

II NOTATION

α_i	Set of parameters defined in body of report.
R, $\dot{R}$	Position and velocity vector.
r, v	Magnitude of R, $\dot{R}$ respectively.
R_0, $\dot{R}_0$, r_0, v_0	The initial values of R, $\dot{R}$, r, v.
$H = R \times \dot{R}$	Angular momentum vector.
h	Magnitude of H.
μ	Gravitational parameter of reference body.
a	Semi-major axis.
S	Differential state transformation matrix (Jacobi matrix).
S^{-1}	Inverse of S.
f, g, $\dot{f}$, $\dot{g}$	Coefficient functions of the two-body problem.
$d = R \cdot \dot{R}$	The dot product of R and $\dot{R}$.
G_n, F_n	Stumpff functions of order n.
α	Generalized Stumpff variables.
β	Regularized differential eccentric anomaly.
Ω	Differential state updating matrix.

III ANALYTIC DEFINITION OF THE MODIFIED NASA VARIABLES

The first three variables consist of three rigid rotations. References 1, 2 and 3 have described similar sets of parameters. The first variable is

(a) A rotation of $R(t)$ about the vector $\dot{R}(t)$ through a small angle α_1. To obtain an explicit expression for $\alpha_1(t)$ as such, consider the expression for $R(t)$ after it has been rotated through the small finite angle $\alpha_1(t)$. Let $R'(t)$ be the resultant value of $R(t)$.

$$R'(t) = R\cos\alpha_1 + \frac{\dot{R}(R\cdot\dot{R})}{v^2}(1-\cos\alpha_1) - \frac{H}{v}\sin\alpha_1 \tag{1}$$

Dot H into the whole equation to obtain

$$\sin\alpha_1 = -\frac{v}{h^2} H\cdot R' \tag{2}$$

Then taking the limit of $\sin\alpha_1$ as α_1 goes to zero gives

$$\lim_{\substack{\alpha_1\to 0\\ R'\to R}} (\sin\alpha_1) = +\alpha_1(t) \tag{3}$$

Equations (2) and (3) give an explicit expression for $\alpha_1(t)$, namely

$$\alpha_1(t) = -\lim_{R'\to R}\left[\frac{v}{h^2} H\cdot R'\right] \tag{4}$$

(b) The second variable is a rotation of the vector $\dot{R}(t)$ about the vector $R(t)$ through a small angle α_2.

$$\alpha_2(t) = + \lim_{\dot{R}' \to \dot{R}} \left[\frac{r}{h^2} H \cdot \dot{R}' \right] \tag{5}$$

(c) The third variable is a rotation of $\dot{R}(t)$ about the angular momentum vector $H = R \times \dot{R}$ through a small angle α_3.

$$\alpha_3(t) = + \lim_{\dot{R}' \to \dot{R}} \left[\frac{1}{hv^2} H \times \dot{R} \cdot \dot{R}' \right] \tag{6}$$

Since neither the magnitude of r nor the dot product $d = R \cdot \dot{R}$ is allowed to vary in this transformation, it is also necessary to rotate R about H through an angle α_3. However, the angle α_3 is determined by the rotation of $\dot{R}$.

(d) The fourth variable,

$$\alpha_4(t) = d = R \cdot \dot{R} \tag{7}$$

$\delta\alpha_4$ is a variation in the scalar function d accomplished by rotating $R(t)$ about H, since a rotation of $\dot{R}$ would change α_3 through a small angle, leaving $\dot{R}(t)$ and the magnitude of the position vector invariant. This is a change in the angle between R and $\dot{R}$ in the orbital plane.

(e) The fifth variable,

$$\alpha_5(t) = \frac{1}{a} = - \frac{\dot{R} \cdot \dot{R}}{\mu} + \frac{2}{[R \cdot R]^{1/2}} \tag{8}$$

A variation in the scalar function $\frac{1}{a}$, i.e. $\delta\alpha_5$, is accomplished by stretching the vector $\dot{R}(t)$, and rotating $R(t)$ in the orbital plane in such a way as to leave the scalar functions r and d unchanged.

(f) The sixth variable

$$\alpha_6(t) = r = [R \cdot R]^{1/2} \tag{9}$$

The variation $\delta\alpha_6$ is a lengthening of $R(t)$. In addition $\dot{R}(t)$ is lengthened in the proper proportion so as to keep $\frac{1}{a}$ invariant and $R(t)$ is rotated in the plane of the orbit in such a way as to keep d unchanged.

(g) In some applications it is useful to consider the earth gravitational constant μ as a seventh state variable.

IV LOCAL STATE TRANSFORMATION MATRICES

It is necessary to be able to transform changes in the cartesian state into changes in the α-state.

(a) The S^{-1} matrix is defined as the point transformation matrix which transforms an infinitesimal vector in the R, $\dot{R}$ space to the α space.

$$\begin{aligned} \Delta\alpha &= S^{-1}\,\Delta X \\ S^{-1} &= \left(\frac{\partial\alpha}{\partial X}\right) \end{aligned} \tag{10}$$

where α is $\alpha_1, \ldots, \alpha_6$ and X represents the $(R, \dot{R})$ vector. Using the definition for α_1 from equation (4) we can differentiate it with respect to any vector L (i.e., with respect to the components of L taken one at a time). In particular, we will let L take on the values of R and $\dot{R}$ respectively.

$$\frac{\partial\alpha_1}{\partial L} = -\lim_{R'\rightarrow R}\left[\frac{v}{h^2}\frac{\partial}{\partial L}(H\cdot R')\right] - \lim_{R'\rightarrow R}\frac{\partial}{\partial L}\left(\frac{v}{h^2}\right) H\cdot R' \tag{11}$$

Taking the limit of the second term of equation (11) zeros it out. At this point we must note that R' is the new position of the R vector which comes from applying the α_1 rotation. Therefore a change in α_1 would change R', but not R in this particular definition and vice versa. So, in differentiating equation (11), we differentiate R' only and then take the limit since the R, $\dot{R}$ system is, in effect, the reference system. Thus equation (11) becomes

$$\frac{\partial \alpha_1}{\partial L} = - \frac{v}{h^2} H \cdot \frac{\partial R}{\partial L} \tag{12}$$

Letting L take on the values R and $\dot{R}$ respectively yields

$$\frac{\partial \alpha_1}{\partial R} = - \frac{v}{h^2} H, \quad \frac{\partial \alpha_1}{\partial \dot{R}} = 0 \tag{13}$$

Using equation (5) as the definition for α_2 we find

$$\frac{\partial \alpha_2}{\partial L} = \lim_{\dot{R}' \to \dot{R}} \left[\frac{r}{h^2} H \cdot \frac{\partial \dot{R}'}{\partial L} \right] \tag{14}$$

Taking the limit yields

$$\frac{\partial \alpha_2}{\partial L} = \frac{r}{h^2} H \cdot \frac{\partial \dot{R}}{\partial L} \tag{15}$$

From which we obtain

$$\frac{\partial \alpha_2}{\partial R} = 0 \quad , \quad \frac{\partial \alpha_2}{\partial \dot{R}} = \frac{r}{h^2} H \tag{16}$$

Similarly

$$\frac{\partial \alpha_3}{\partial L} = \frac{1}{hv^2} H \times \dot{R} \cdot \frac{\partial \dot{R}}{\partial L} \tag{17}$$

so that

$$\frac{\partial \alpha_3}{\partial R} = 0 \ , \quad \frac{\partial \alpha_3}{\partial \dot{R}} = \frac{1}{h v^2} \quad H \times \dot{R} \quad . \tag{18}$$

Using equation (7) for $\alpha_4(t)$ we have

$$\frac{\partial \alpha_4}{\partial R} = \dot{R} \quad , \quad \frac{\partial \alpha_4}{\partial \dot{R}} = R \quad . \tag{19}$$

Differentiating equation (8) to get the fifth row of the S^{-1} matrix yields

$$\frac{\partial \alpha_5}{\partial L} = - \frac{2 \dot{R} \cdot \frac{\partial \dot{R}}{\partial L}}{\mu} - \frac{2}{r^3} R \cdot \frac{\partial R}{\partial L} + \frac{\dot{R} \cdot \dot{R}}{\mu^2} \frac{\partial \mu}{\partial L} \tag{20}$$

This gives

$$\frac{\partial \alpha_5}{\partial R} = - \frac{2}{r^3} R \ , \quad \frac{\partial \alpha_5}{\partial \dot{R}} = - \frac{2}{\mu} \dot{R}$$

If μ is used as a seventh variable then:

$$\frac{\partial \alpha_5}{\partial \mu} = \frac{v^2}{\mu^2} \tag{21}$$

The sixth row of the S^{-1} matrix consists of the partials of α_6 with respect to the elements of R and $\dot{R}$ respectively.

$$\alpha_6 = [R \cdot R]^{1/2}$$

$$\frac{\partial \alpha_6}{\partial L} = \frac{1}{2} [R \cdot R]^{-1/2} \left[2 R \cdot \frac{\partial R}{\partial L} \right] = \frac{R}{r} \cdot \frac{\partial R}{\partial L} \tag{22}$$

So

$$\frac{\partial \alpha_6}{\partial R} = \frac{R}{r} \quad , \quad \frac{\partial \alpha_6}{\partial \dot{R}} = 0 \quad . \tag{23}$$

The seventh row:

$\mu = \mu$ and is not a function of R and $\dot{R}$.

Hence six zeros and 1 on the diagonal.

Collecting the results of equations (13), (16), (18), (19), (21) and (23) gives us the S^{-1} matrix.

$$S^{-1} = \begin{bmatrix} \frac{-v}{h^2} H & 0 \\ 0 & \frac{r}{h^2} H \\ 0 & \frac{1}{hv^2} H X \dot{R} \\ \dot{R} & R \\ \frac{-2}{r^3} R & \frac{-2}{\mu} \dot{R} \\ \frac{1}{r} R & 0 \end{bmatrix} \tag{24}$$

Here H, R, $\dot{R}$ are considered row vectors. If μ is used as a seventh state variable an augmented state transformation matrix results:

$$S_7^{-1} = \begin{pmatrix} \begin{bmatrix} \\ & & S_6^{-1} & & \\ \\ \end{bmatrix} & \begin{matrix} 0 \\ 0 \\ 0 \\ 0 \\ v^2/\mu^2 \\ 0 \end{matrix} \\ 0\ 0\ 0\ 0\ 0\ 0 & 1 \end{pmatrix}$$

(b) The S matrix is a 6 X 6 point transformation matrix representing the partials of x through $\dot{z}$ and μ with respect to α_1 through α_6 and μ. Symbolically it is written

$$\Delta \alpha = S \, \Delta x$$

$$S(t) = \left(\frac{\partial x(t)}{\partial \alpha(t)} \right) \tag{25}$$

Since α_1, α_2 and α_3 are three rigid rotations of R about $\dot{R}$, $\dot{R}$ about R, and R and $\dot{R}$ about H, respectively, then the partial derivatives of any vector L with respect to α_1, α_2 and α_3 are given immediately by

$$\frac{\partial L}{\partial \alpha_1} = \frac{\dot{R}}{v} \times L \tag{26}$$

$$\frac{\partial L}{\partial \alpha_2} = \frac{R}{r} \times L \tag{27}$$

$$\frac{\partial L}{\partial \alpha_3} = \frac{H}{h} \times L \tag{28}$$

Replacing L by R and $\dot{R}$ respectively in the above equations gives the first three columns of the S matrix. Each equation produces one column.

$$\begin{aligned} &\frac{\partial R}{\partial \alpha_1} = -\frac{H}{v}, \quad &&\frac{\partial R}{\partial \alpha_2} = 0, \quad &&\frac{\partial R}{\partial \alpha_3} = \frac{1}{h} H \times R \\ &\frac{\partial \dot{R}}{\partial \alpha_1} = 0, \quad &&\frac{\partial \dot{R}}{\partial \alpha_2} = \frac{H}{r}, \quad &&\frac{\partial \dot{R}}{\partial \alpha_3} = \frac{1}{h} H \times \dot{R} \end{aligned} \tag{29}$$

α_1 and α_2 define the orbit plane; hence changes in $\alpha_{3,4,5,6}$ must be in the orbit plane and thus the partial derivatives must be representable as linear combinations of R and $\dot{R}$. Further α_3 is measured by a rotation of $\dot{R}$ about H, hence the partial derivatives of $\dot{R}$ with respect to $\alpha_{4,5,6}$ must not entail any change in the direction of $\dot{R}$.

The general form for columns 4, 5 and 6 of the S matrix then must be:

$$\frac{\partial R}{\partial \alpha_j} = c_{1j} R + C_{2j} H \times R \qquad j = 4, 5, 6 \tag{30}$$

$$\frac{\partial \dot{R}}{\partial \alpha_j} = C_{3j} \dot{R} \tag{31}$$

This general form permits a lengthening of R and $\dot{R}$ and a change in the angle between them by rotating R about H . The requirement that the α's be independent orbit parameters yields

$$\frac{\partial \alpha_i(t)}{\partial \alpha_j(t)} = \delta_{ij} \quad , \tag{32}$$

the Kronecker delta which equals 1 for $i = j$ and equals zero for $i \neq j$.

The method used to obtain the last three columns of the S matrix is to take the definitions of alphas 4, 5, and 6 from equations (7), (8) and (9) and differentiate them with respect to α_j and then to substitute equations (30), (31) and (32) into the equations in order to solve for the C_{1j}, C_{2j} and C_{3j} coefficients.

$$\frac{\partial \alpha_4}{\partial \alpha_j} = \dot{R} \cdot \frac{\partial R}{\partial \alpha_j} + R \cdot \frac{\partial \dot{R}}{\partial \alpha_j} \tag{33}$$

$$\frac{\partial \alpha_5}{\partial \alpha_j} = -\frac{2}{\mu} \dot{R} \cdot \frac{\partial \dot{R}}{\partial \alpha_j} - \frac{2}{r^3} R \cdot \frac{\partial R}{\partial \alpha_j} + \frac{v^2}{\mu^2} \frac{\partial \mu}{\partial \alpha_j} \tag{34}$$

$$\frac{\partial \alpha_6}{\partial \alpha_j} = \frac{1}{r} R \cdot \frac{\partial R}{\partial \alpha_j} \tag{35}$$

Making the substitutions for $\frac{\partial R}{\partial \alpha_j}$ and $\frac{\partial \dot{R}}{\partial \alpha_j}$ gives us three equations with three unknown C's for each value of $j = 4$, 5 or 6. Namely

$$(C_{1j} + C_{3j}) d + C_{2j} h^2 = \delta_{4j} \tag{36}$$

$$\frac{-2\,C_{3j}\,v^2}{\mu} - \frac{2\,C_{1j}}{r} = \delta_{5j} - \frac{v^2}{\mu^2}\,\delta_{7j} \tag{37}$$

$$C_{1j}\,r = \delta_{6j} \quad . \tag{38}$$

From equation (38) we can see that $C_{14} = C_{15} = 0$, $C_{17} = 0$ and that

$$C_{16} = \frac{1}{r} \quad . \tag{39}$$

Using $C_{14} = 0$, equation (37) gives

$$\frac{-2\,C_{3j}\,v^2}{\mu} = 0 \quad . \tag{40}$$

So $C_{34} = 0$.

Now equation (36) gives

$$C_{24}\,h^2 = 1 \tag{41}$$

or

$$C_{24} = \frac{1}{h^2} \quad . \tag{42}$$

So the fourth column of the S matrix is

$$\begin{aligned} \frac{\partial R}{\partial \alpha_4} &= \frac{1}{h^2}\ H \times R \\ \frac{\partial \dot{R}}{\partial \alpha_4} &= 0 \end{aligned} \tag{43}$$

Using the fact that $C_{15} = 0$, equation (37) gives

$$C_{35} = - \frac{\mu}{2 v^2} \quad . \tag{44}$$

And equation (36) gives

$$- \frac{\mu d}{2 v^2} + C_{25} h^2 = 0 \tag{45}$$

$$C_{25} = \frac{\mu d}{2 v^2 h^2} \quad . \tag{46}$$

So the fifth column of the S matrix is

$$\begin{aligned} \frac{\partial R}{\partial \alpha_5} &= \frac{\mu d}{2v^2 h^2} \quad H \times R \\ \frac{\partial \dot{R}}{\partial \alpha_5} &= - \frac{\mu}{2 v^2} \quad \dot{R} \quad . \end{aligned} \tag{47}$$

Finally from $C_{16} = \frac{1}{r}$ and equation (37) we have

$$\frac{-2 C_{36} v^2}{\mu} - \frac{2}{r^2} = 0 \tag{48}$$

$$C_{36} = - \frac{\mu}{r^2 v^2} \quad . \tag{49}$$

And equation (36) gives

$$\left(\frac{1}{r} - \frac{\mu}{r^2 v^2} \right) d + C_{26} h^2 = 0 \tag{50}$$

$$C_{26} = - \frac{\mu d}{r^2 v^2 h^2} \left(\frac{r v^2}{\mu} - 1 \right) \tag{51}$$

So the sixth column of the S matrix has the components:

$$\begin{aligned} \frac{\partial R}{\partial \alpha_6} &= \frac{1}{r} R - \frac{\mu d \left(\frac{r v^2}{\mu} - 1 \right)}{r^2 v^2 h^2} H \times R \\ \frac{\partial \dot{R}}{\partial \alpha_6} &= - \frac{\mu}{r^2 v^2} \dot{R} \end{aligned} \tag{52}$$

For seventh column, from equation (38) we have $C_{17} = 0$, from equation (37) $C_{37} = \frac{1}{2\mu}$, and from (36) $\frac{d}{2\mu} = - C_{27} h^2$

$C_{27} = - \frac{d}{2 \mu h^2}$. So the seventh column of the S matrix is

$$\begin{bmatrix} \frac{-d}{2 \mu h^2} H \times R \\ \frac{1}{2 \mu} \dot{R} \end{bmatrix}$$

Collecting the results of equations (29), (43), (47) and (52) gives the S matrix:

$$S = \begin{bmatrix} \frac{-H}{v} & 0 & \frac{1}{h} H \times R & \frac{1}{h^2} H \times R & \frac{\mu d}{2 v^2 h^2} H \times R & \frac{R}{r} - \frac{\mu d \left(\frac{r v^2}{\mu} - 1 \right)}{r^2 v^2 h^2} H \times R \\ 0 & \frac{H}{r} & \frac{1}{h} H \times \dot{R} & 0 & \frac{-\mu}{2 v^2} \dot{R} & \frac{-\mu}{r^2 v^2} \dot{R} \end{bmatrix} \tag{53}$$

Here H, R, $\dot{R}$ are considered column vectors.

Finally, it is a simple exercise to show that the product $S^{-1}S$ does in fact equal the unit matrix.

If μ is considered as a seventh variable

$$S_7 = \begin{bmatrix} S_6 & \dfrac{-d}{2\,\mu h^2} H \times R \\ & + \dfrac{1}{2\,\mu}\,\dot{R} \\ 0 & 1 \end{bmatrix}$$

V UNIVERSAL FORMULATION OF THE KEPLER PROBLEM

The orbit parameters described in the foregoing sections are (with the exception of α_5) not constants of the motion, hence equations must be developed which relate changes in α at time t_0 with changes at time t. These relations are conveniently derived from the universal solution obtained by K. Stumpff and first published in Reference 4. It is used as a basis for the form of the solution described below. Other forms of this original solution have been used by Herrick (Reference 5), Boyce (Reference 2) and the present authors (Reference 3). It should be noted that this solution is also valid for rectilinear orbits.

The solution of the two-body problem, in Cartesian coordinates, is given as a function of the initial conditions as follows:

$$R = f\, R_0 + g\, \dot{R}_0 \tag{54}$$

$$\dot{R} = \dot{f}\, R_0 + \dot{g}\, \dot{R}_0 \tag{55}$$

The functions f, g, $\dot{f}$ and $\dot{g}$ are given in terms of the initial conditions and the increment of time from the initial time, $t - t_0$, as follows:

$$f = 1 - \frac{G_2}{r_0} \tag{56}$$

$$g = \frac{r_0}{\sqrt{\mu}} G_1 + \frac{d_0}{\mu} G_2 = t - t_0 - \frac{G_3}{\sqrt{\mu}} = \frac{r\, G_1}{\sqrt{\mu}} - \frac{d}{\mu} G_2 \tag{57}$$

$$\dot{f} = - \frac{\sqrt{\mu}}{r\, r_0} G_1 \tag{58}$$

$$\dot{g} = 1 - \frac{G_2}{r} = \frac{1}{r} \left(r_0 G_0 + \frac{d_0}{\sqrt{\mu}} G_1\right) \tag{59}$$

The functions r_0, d_0, v_0, r and d are defined as

$$r_0 = \left[R(t_0) \cdot R(t_0)\right]^{1/2} \tag{60}$$

$$d_0 = R(t_0) \cdot \dot{R}(t_0) \tag{61}$$

$$v_0 = \left[\dot{R}(t_0) \cdot \dot{R}(t_0)\right]^{1/2} \tag{62}$$

$$r = \left[R(t) \cdot R(t)\right]^{1/2} = G_2 + r_0 G_0 + \frac{d_0}{\sqrt{\mu}} G_1 \tag{63}$$

$$d = R(t) \cdot \dot{R}(t) = \sqrt{\mu}\left(1 - \frac{r_0}{a}\right) G_1 + d_0 G_0 \tag{64}$$

where

$$\frac{1}{a}(t) = \frac{1}{a}(t_0) = \frac{2}{r_0} - \frac{v_0^2}{\mu} \tag{65}$$

The functions G_n are defined as

$$G_n = \beta^n F_n(\alpha) \tag{66}$$

where

$$F_n(\alpha) = \sum_{k=0}^{\infty} \frac{(-\alpha)^k}{(2k+n)!} \tag{67}$$

$$\beta = \sqrt{a}\ \theta = \sqrt{a}\,(E - E(t_0)) \tag{68}$$

$$\alpha = \theta^2 = \frac{\beta^2}{a} \tag{69}$$

The variable β is the regularization parameter used to unify the hyperbolic, elliptic and parabolic cases. It is noted that β is always real since the eccentric anomaly becomes imaginary whenever the semi-major axis becomes negative.

The functions $F_n(\alpha)$ are in reality the sine and cosine series with a finite number of initial terms removed and with some power of α factored out so that they all start with a constant term. For the hyperbolic case, α is negative, and the $F_n(\alpha)$ are the hyperbolic sine and cosine series. In the parabolic case, $\alpha = 0$, so that each $F_n(\alpha)$ series reduces to merely its constant term. To obtain the universal anomaly, β, from the increment in time, it is necessary to solve Kepler's equation given below

$$\begin{aligned} \sqrt{\mu}\,(t-t_0) &= G_3 + r_0\, G_1 + \frac{d_0}{\sqrt{\mu}}\, G_2 \\ &= \beta^3 F_3 + r_0\, \beta\, F_1 + \frac{d_0}{\sqrt{\mu}}\, \beta^2 F_2 \end{aligned} \tag{70}$$

Once β is determined, one merely has to form the necessary G_n's and evaluate the formulas for f, g, $\dot{f}$ and $\dot{g}$ and the two-body system is complete.

VI THE DEVELOPMENT OF THE STATE TRANSITION MATRIX FOR THE MODIFIED NASA VARIABLES

The state transition matrix is defined as follows:

$$\boldsymbol{\Omega}\,(t,\ t_0) = \frac{\partial \alpha\,(t)}{\partial \alpha\,(t_0)} \tag{71}$$

Using equations (12), (15) and (17) which were developed for the purpose of obtaining the S^{-1} matrix, we can obtain the first three rows of the $\boldsymbol{\Omega}$ matrix. These equations yield

$$\frac{\partial \alpha_1\,(t)}{\partial \alpha_j\,(0)} = -\frac{v}{h^2}\ H \cdot \frac{\partial R}{\partial \alpha_j\,(0)} \tag{72}$$

$$\frac{\partial \alpha_2\,(t)}{\partial \alpha_j\,(0)} = \frac{r}{h^2}\ H \cdot \frac{\partial \dot{R}}{\partial \alpha_j\,(0)} \tag{73}$$

$$\frac{\partial \alpha_3\,(t)}{\partial \alpha_j\,(0)} = \frac{1}{h v^2}\ H \times \dot{R} \cdot \frac{\partial \dot{R}}{\partial \alpha_j\,(0)} \tag{74}$$

To obtain the first row of the $\boldsymbol{\Omega}$ matrix one needs only to evaluate $\frac{\partial R}{\partial \alpha_j\,(0)}$

$$\frac{\partial R}{\partial \alpha_j\,(0)} = \frac{\partial}{\partial \alpha_j\,(0)} \left[f\,R_0 + g\,\dot{R}_0 \right]$$

$$= f\,\frac{\partial R_0}{\partial \alpha_j\,(0)} + g\,\frac{\partial \dot{R}_0}{\partial \alpha_j\,(0)} + \frac{\partial f}{\partial \alpha_j\,(0)}\,R_0 + \frac{\partial g}{\partial \alpha_j\,(0)}\,\dot{R}_0 \tag{75}$$

Substituting equation (75) into (72) yields

$$\frac{\partial \alpha_1 (t)}{\partial \alpha_j (0)} = - \frac{v}{h^2} H \cdot \left(f \frac{\partial R_0}{\partial \alpha_j (0)} + g \frac{\partial \dot{R}_0}{\partial \alpha_j (0)} \right) \tag{76}$$

The last two terms of (75) dropped out because in the two-body system $H = H_0$. Now $\frac{\partial R_0}{\partial \alpha_j (0)}$ and $\frac{\partial \dot{R}_0}{\partial \alpha_j (0)}$ are the elements of the $S(t_0)$ matrix [see equation (53) and imagine zero subscripts throughout]. It can be seen clearly that H dotted into $H \times R_0$, R_0, $H \times \dot{R}_0$, and $\dot{R}_0$ are all zero so that the 3rd through 6th items of the first row are all zero.

So equation (76) yields

$$\frac{\partial \alpha_1 (t)}{\partial \alpha_1 (0)} = \frac{v}{v_0} f , \quad \frac{\partial \alpha_1 (t)}{\partial \alpha_2 (0)} = - \frac{v}{r_0} g , \quad \frac{\partial \alpha_1 (t)}{\partial \alpha_j (0)} = 0 \text{ for } j = 3, 4, 5, 6, 7 \tag{77}$$

Returning to equations (73) and (74) requires $\frac{\partial \dot{R}}{\partial \alpha_j (0)}$

$$\begin{aligned} \frac{\partial \dot{R}}{\partial \alpha_j (0)} &= \dot{f} \frac{\partial R_0}{\partial \alpha_j (0)} + \dot{g} \frac{\partial \dot{R}_0}{\partial \alpha_j (0)} + \frac{\partial \dot{f}}{\partial \alpha_j (0)} R_0 + \frac{\partial \dot{g}}{\partial \alpha_j (0)} \dot{R}_0 \\ &= \dot{f} \frac{\partial R_0}{\partial \alpha_j (0)} + \dot{g} \frac{\partial \dot{R}_0}{\partial \alpha_j (0)} + \frac{\partial f}{\partial \alpha_j (0)} (\dot{g} R - g \dot{R}) + \frac{\partial \dot{g}}{\partial \alpha_j (0)} (- \dot{f} R + f \dot{R}) \end{aligned} \tag{78}$$

Substituting equation (78) into (73) yields

$$\frac{\partial \alpha_2 (t)}{\partial \alpha_j (0)} = \frac{r}{h^2} H \cdot \left(\dot{f} \frac{\partial R_0}{\partial \alpha_j (0)} + \dot{g} \frac{\partial \dot{R}_0}{\partial \alpha_j (0)} \right) \tag{79}$$

Making use of the $S(t_0)$ matrix, equation (79) gives the second row of the Ω matrix, namely

$$\frac{\partial \alpha_2(t)}{\partial \alpha_1(0)} = -\frac{r}{v_0}\dot{f}, \quad \frac{\partial \alpha_2(t)}{\partial \alpha_2(0)} = \frac{r}{r_0}\dot{g}, \quad \frac{\partial \alpha_2(t)}{\partial \alpha_j(0)} = 0 \text{ for } j = 3, 4, 5, 6, 7 \tag{80}$$

To obtain the third row of the Ω matrix substitute equation (78) into equation (74). This yields

$$\frac{\partial \alpha_3(t)}{\partial \alpha_j(0)} = \frac{1}{hv^2} H \, X \, \dot{R} \cdot \left\{ \dot{f} \frac{\partial R_0}{\partial \alpha_j(0)} + \dot{g} \frac{\partial \dot{R}_0}{\partial \alpha_j(0)} + R \left(\dot{g} \frac{\partial \dot{f}}{\partial \alpha_j(0)} - \dot{f} \frac{\partial \dot{g}}{\partial \alpha_j(0)} \right) \right\} \tag{81}$$

In evaluating equation (81) with respect to $\alpha_1(0)$, $\alpha_2(0)$ and $\alpha_3(0)$ the right hand term drops out since $\dot{f}$ and $\dot{g}$ are functions of $\alpha_4(0)$, $\alpha_5(0)$ and $\alpha_6(0)$ only. The third term is

$$\frac{\partial \alpha_3(t)}{\partial \alpha_3(0)} = \frac{1}{hv^2} H \, X \, \dot{R} \cdot \left(\frac{\dot{f}}{h} H \, X \, R_0 + \frac{\dot{g}}{h} H \, X \, \dot{R}_0 \right)$$

$$= \frac{1}{h^2 v^2} H \, X \, \dot{R} \cdot H \, X \left(\dot{f} \, R_0 + \dot{g} \, \dot{R}_0 \right)$$

$$= \frac{1}{h^2 v^2} H \, X \, \dot{R} \cdot H \, X \, \dot{R}$$

$$= 1 \tag{82}$$

So for the first three terms equation (81) gives

$$\frac{\partial \alpha_3(t)}{\partial \alpha_1(0)} = 0 \, , \quad \frac{\partial \alpha_3(t)}{\partial \alpha_2(0)} = 0 \, , \quad \frac{\partial \alpha_3(t)}{\partial \alpha_3(0)} = 1 \quad . \tag{83}$$

The last three terms of the third row will be evaluated from equation (81) after the rest of the Ω matrix has been formed.

The method for deriving the last three rows of the Ω matrix is fairly straightforward. It amounts to expressing $\alpha_4(t)$, $\alpha_5(t)$ and $\alpha_6(t)$ in terms of $\alpha_4(0)$, $\alpha_5(0)$, $\alpha_6(0)$ and the G_n's . In order to differentiate these expressions with respect to $\alpha_j(0)$ one needs $\dfrac{\partial G_n}{\partial \alpha_j(0)}$.

From equation (66)

$$G_n = \beta^n F_n(\alpha) = \beta^n F_n(\theta^2)$$

$$= \beta^n \left[\frac{1}{n!} - \frac{\theta^2}{(n+2)!} + \frac{\theta^4}{(n+4)!} - \dots \right]$$

$$= \frac{\beta^n}{\theta^n} \left[\frac{\theta^n}{n!} - \frac{\theta^{n+2}}{n+2!} + \frac{\theta^{n+4}}{(n+4)!} - \dots \right] \quad . \tag{84}$$

Differentiating the above equation gives

$$\frac{\partial G_n}{\partial \alpha_j(0)} \quad n \beta^{n-1} F_n \frac{\partial \beta}{\partial \alpha_j(0)} + \frac{\beta^n}{\theta^n} \left[\frac{n \, \theta^{n-1}}{n(n-1)!} - \frac{(n+2) \, \theta^{n+1}}{(n+2)(n+1)!} + \dots \right] \frac{\partial \theta}{\partial \alpha_j(0)}$$

$$- n \frac{\beta^n}{\theta^{n+1}} \left[\frac{\theta^n}{n!} - \frac{\theta^{n+2}}{(n+2)!} + \dots \right] \frac{\partial \theta}{\partial \alpha_j(0)}$$

$$= n \beta^{n-1} F_n \frac{\partial \beta}{\partial \alpha_j(0)} + \beta^n \left[\frac{F_{n-1}}{\theta} - \frac{n F_n}{\theta} \right] \frac{\partial \theta}{\partial \alpha_j(0)} \quad . \tag{85}$$

From

$$\theta^2 = \frac{\beta^2}{a}$$

we have

$$\theta \frac{\partial \theta}{\partial \alpha_j(0)} = \frac{\beta}{a} \frac{\partial \beta}{\partial \alpha_j(0)} + \frac{1}{2} \beta^2 \frac{\partial(\frac{1}{a})}{\partial \alpha_j(0)} \tag{86}$$

So equation (85) becomes

$$\frac{\partial G_n}{\partial \alpha_j(0)} = \frac{\partial \beta}{\partial \alpha_j(0)} \left[n \beta^{n-1} F_n + \frac{\beta^{n+1}}{a \theta^2} F_{n-1} - \frac{n \beta^{n+1}}{a \theta^2} F_n \right]$$

$$+ \frac{\beta^{n+2}}{2 \theta^2} \frac{\partial(\frac{1}{a})}{\partial \alpha_j(0)} \left[F_{n-1} - n F_n \right]$$

$$= \beta^{n-1} F_{n-1} \frac{\partial \beta}{\partial \alpha_j(0)}$$

$$+ \frac{\beta^{n+2}}{2 \theta^2} \frac{\partial(\frac{1}{a})}{\partial \alpha_j(0)} \left[\frac{1}{(n-1)!} - \frac{\theta^2}{(n+1)!} + \frac{\theta^4}{(n+3)!} \cdots \right.$$

$$\left. - \frac{n}{n!} + \frac{n \theta^2}{(n+2)!} - \frac{n \theta^4}{(n+4)!} + \cdots \right]$$

$$= G_{n-1} \frac{\partial \beta}{\partial \alpha_j(0)} + \frac{\beta^{n+2}}{2} \frac{\partial(\frac{1}{a})}{\partial \alpha_j(0)} \left[n F_{n+2} - F_{n+1} \right]$$

$$= G_{n-1} \frac{\partial \beta}{\partial \alpha_j(0)} + \frac{1}{2} \left[n G_{n+2} - \beta G_{n+1} \right] \frac{\partial(\frac{1}{a})}{\partial \alpha_j(0)} \tag{87}$$

In order to obtain $\dfrac{\partial \beta}{\partial \alpha_j(0)}$ differentiate Kepler's equation, which is of the form

$$\sqrt{\mu}\ \Delta t = G_3 + r_0 G_1 + \frac{d_0}{\sqrt{\mu}} G_2 \quad . \tag{88}$$

The differentiation of equation (88) making use of equation (87) yields

$$\begin{aligned} \frac{1}{2} \frac{\Delta t}{\sqrt{\mu}} \frac{\partial \mu}{\partial \alpha_j(0)} &= \frac{\partial \beta}{\partial \alpha_j(0)} \left(G_2 + r_0 G_0 + \frac{d_0}{\sqrt{\mu}} G_1 \right) \\ &+ \frac{1}{2} \frac{\partial \frac{1}{a}}{\partial \alpha_j(0)} \left(3 G_5 - \beta G_4 + r_0 G_3 - r_0 \beta G_2 + 2 \frac{d_0}{\sqrt{\mu}} G_4 - \frac{d_0}{\sqrt{\mu}} \beta G_3 \right) \\ &+ G_1 \frac{\partial r_0}{\partial \alpha_j(0)} + \frac{G_2}{\sqrt{\mu}} \frac{\partial d_0}{\partial \alpha_j(0)} - \frac{1}{2} \frac{d_0}{\mu^{3/2}} G_2 \frac{\partial \mu}{\partial \alpha_j(0)} \end{aligned} \tag{89}$$

The coefficient of $\dfrac{\partial \beta}{\partial \alpha_j(0)}$ in equation (89) is r. Finally we have

$$\begin{aligned} \frac{\partial \beta}{\partial \alpha_j(0)} &= - \frac{1}{2r} \left(3G_5 - \beta G_4 + r G_3 - \sqrt{\mu}\, \Delta t\, G_2 \right) \frac{\partial \frac{1}{a}}{\partial \alpha_j(0)} \\ &- \frac{G_1}{r} \frac{\partial r_0}{\partial \alpha_j(0)} \\ &- \frac{G_2}{r\sqrt{\mu}} \frac{\partial d_0}{\partial \alpha_j(0)} \\ &+ \frac{1}{2\mu r} \left(\sqrt{\mu}\, \Delta t + \frac{d_0}{\sqrt{\mu}} G_2 \right) \frac{\partial \mu}{\partial \alpha_j(0)} \end{aligned} \tag{90}$$

One other useful identity is obtained by making use of the definition of G_n:

$$G_n = \frac{\beta^n}{n!} - \frac{1}{a} G_{n+2} \quad . \tag{91}$$

This may be used to obtain

$$G_{-1} = -\frac{1}{a} G_1 \quad . \tag{92}$$

The equations for $\alpha_4(t)$, $\alpha_5(t)$ and $\alpha_6(t)$ which are to be differentiated in order to obtain the 4^{th}, 5^{th} and 6^{th} rows respectively of the Ω matrix are as follows:

$$\alpha_4(t) = d = \sqrt{\mu}\left(1 - \frac{r_0}{a}\right) G_1 + d_0 G_0 \tag{93}$$

$$\alpha_5(t) = \frac{1}{a} = \frac{1}{a_0} = \alpha_5(0) \tag{94}$$

$$\alpha_6(t) = r = G_2 + r_0 G_0 + \frac{d_0}{\sqrt{\mu}} G_1 \quad . \tag{95}$$

Making use of equation (87) for differentiating G_n we have, starting with equation (93),

$$\begin{aligned}
\frac{\partial \alpha_4(t)}{\partial \alpha_j(0)} &= \left[\sqrt{\mu}\left(1 - \frac{r_0}{a}\right) G_0 - \frac{d_0}{a} G_1\right] \frac{\partial \beta}{\partial \alpha_j(0)} \\
&+ \frac{1}{2} \frac{\partial \frac{1}{a}}{\partial \alpha_j(0)} \left[\sqrt{\mu}\left(1 - \frac{r_0}{a}\right)\left(G_3 - \beta G_2\right) - d_0 \beta G_1\right] \\
&- \frac{\sqrt{\mu}}{a} G_1 \frac{\partial r_0}{\partial \alpha_j(0)} + G_0 \frac{\partial d_0}{\partial \alpha_j(0)} - \sqrt{\mu}\, r_0 G_1 \frac{\partial \frac{1}{a}}{\partial \alpha_j(0)} \\
&+ \frac{1}{2\sqrt{\mu}}\left(1 - \frac{r_0}{a}\right) G_1 \frac{\partial \mu}{\partial \alpha_j(0)}
\end{aligned} \tag{96}$$

From equation (91)

$$G_0 = 1 - \frac{1}{a} G_2 \quad , \quad G_1 = \beta - \frac{1}{a} G_3 \tag{97}$$

So the coefficient of $\dfrac{\partial \beta}{\partial \alpha_j(0)}$ is

$$\sqrt{\mu} \left(1 - \frac{1}{a} G_2 \right) - \frac{\sqrt{\mu}\, r_0}{a} G_0 - \frac{d_0}{a} G_1$$

$$= \sqrt{\mu} - \frac{\sqrt{\mu}}{a} \left(G_2 + r_0 G_0 + \frac{d_0}{\sqrt{\mu}} G_1 \right)$$

$$= \sqrt{\mu} \left(1 - \frac{r}{a} \right)$$

And the coefficient of $\dfrac{\partial \frac{1}{a}}{\partial a_j (0)}$ in equation (96) is

$$- \frac{\sqrt{\mu}}{2} \beta \left(r - r_0 \right) + \frac{\sqrt{\mu}}{2} \left(G_3 - \frac{r_0}{a} G_3 \right) - \sqrt{\mu}\, r_0 G_1$$

$$= - \frac{\sqrt{\mu}}{2} \left[\beta r - \beta r_0 - G_3 + \beta r_0 - r_0 G_1 + 2 r_0 G_1 \right]$$

$$= \frac{\sqrt{\mu}}{2} \left[G_3 - \beta r - r_0 G_1 \right]$$

So equation (96) becomes

$$\frac{\partial \alpha_4(t)}{\partial \alpha_j(0)} = \left[- \frac{\sqrt{\mu}\, G_1}{r} \left(1 - \frac{r}{a} \right) - \frac{\sqrt{\mu}}{a} G_1 \right] \frac{\partial r_0}{\partial \alpha_j(0)}$$

$$+ \left[- \frac{G_2}{r} \left(1 - \frac{r}{a} \right) + G_0 \right] \frac{\partial d_0}{\partial \alpha_j(0)}$$

$$+ \left[- \frac{\sqrt{\mu}}{2r} \left(1 - \frac{r}{a} \right) \left(3\, G_5 - \beta\, G_4 + r\, G_3 - \sqrt{\mu}\, \Delta t\, G_2 \right) \right.$$

$$\left. + \frac{\sqrt{\mu}}{2} \left(G_3 - \beta\, r - r_0\, G_1 \right) \right] \frac{\partial \frac{1}{a}(0)}{\partial \alpha_j(0)} \tag{98}$$

$$+ \left[\frac{1}{2\sqrt{\mu}} \left(1 - \frac{r_0}{a} \right) G_1 + \frac{1}{2\sqrt{\mu}\, r} \left(1 - \frac{r}{a} \right) \left(\sqrt{\mu}\, \Delta t + \frac{d_0}{\sqrt{\mu}} G_2 \right) \right] \frac{\partial \mu(0)}{\partial \alpha_j(0)}$$

Equation (98) reduces directly to

$$\frac{\partial \alpha_4(t)}{\partial \alpha_j(0)} = r_0\, \dot{f}\, \frac{\partial r_0}{\partial \alpha_j(0)} + \dot{g}\, \frac{\partial d_0}{\partial \alpha_j(0)}$$

$$+ \frac{\partial \frac{1}{a}}{\partial \alpha_j(0)} \left[- \frac{3}{2}\, \mu\, \Delta t - \frac{1}{2}\, \mu\, g\, (\dot{g} - 2) + \frac{\sqrt{\mu}}{2r} \left(\beta\, G_4 - 3\, G_5 + G_2 G_3 \right) \right]$$

$$+ \frac{1}{2\mu} \left[d - d_0\, \dot{g} + \frac{\mu\, \Delta t}{r} \left(\frac{r v^2}{\mu} - 1 \right) \right] \frac{\partial \mu(0)}{\partial \alpha_j(0)} \tag{99}$$

which in turn yields the fourth row of the Ω matrix.

The differentiation of equation (94) yields

$$\Omega_{5,5} = 1 \tag{100}$$

and all other elements of the fifth row equal to zero.

The sixth row of the Ω matrix represents the partials of $r(t)$ with respect to all of the alphas at t_0. Differentiating the expression for r given by equation (95) with respect to any $\alpha_j(0)$ yields

$$\begin{aligned}
\frac{\partial \alpha_6(t)}{\partial \alpha_j(0)} &= \left[G_1 - \frac{r_0}{a} G_1 + \frac{d_0}{\sqrt{\mu}} G_0 \right] \frac{\partial \beta}{\partial \alpha_j(0)} \\
&+ \frac{1}{2} \frac{\partial (\frac{1}{a})}{\partial \alpha_j(0)} \left[2\, G_4 - \beta\, G_3 - r_0 \beta\, G_1 + \frac{d_0}{\sqrt{\mu}} \left(G_3 - \beta\, G_2 \right) \right] \\
&+ G_0 \frac{\partial r_0}{\partial \alpha_j(0)} + \frac{1}{\sqrt{\mu}} G_1 \frac{\partial d_0}{\partial \alpha_j(0)} - \frac{1}{2} \frac{d_0}{\mu \sqrt{\mu}} G_1 \frac{\partial \mu(0)}{\partial \alpha_j(0)} .
\end{aligned} \tag{101}$$

The coefficient of $\frac{\partial \beta}{\partial \alpha_j(0)}$ in the above equation is $\frac{d}{\sqrt{\mu}}$. And the coefficient of $\frac{\partial \frac{1}{a}}{\partial \alpha_j(0)}$ in equation (101) is

$$G_4 - \frac{1}{2} \beta \sqrt{\mu} \Delta t + \frac{1}{2} \frac{d_0}{\sqrt{\mu}} G_3 \quad .$$

Making use of these two coefficients and the expression for $\frac{\partial \beta}{\partial \alpha_j(0)}$ from equation (90) gives

$$\frac{\partial \alpha_6(t)}{\partial \alpha_j(0)} = \left[G_0 - \frac{d}{\sqrt{\mu}} \frac{G_1}{r} \right] \frac{\partial r_0}{\partial \alpha_j(0)}$$

$$+ \frac{1}{\sqrt{\mu}} \left[G_1 - \frac{d\, G_2}{\sqrt{\mu}\, r} \right] \frac{\partial d_0}{\partial \alpha_j(0)}$$

$$+ \frac{\partial \frac{1}{a}}{\partial \alpha_j(0)} \left[G_4 - \frac{1}{2} \beta \sqrt{\mu}\, \Delta t + \frac{1}{2} \frac{d_0}{\sqrt{\mu}} G_3 \right.$$

$$\left. - \frac{d}{2\sqrt{\mu}\, r} \left(3\, G_5 - \beta G_4 + r\, G_3 - \sqrt{\mu}\, \Delta t\, G_2 \right) \right]$$

$$+ \frac{\partial \mu(0)}{\partial \alpha_j(0)} \left[\frac{d}{2\,\mu \sqrt{\mu}\, r} \left(\sqrt{\mu}\, \Delta t + \frac{d_0}{\sqrt{\mu}} G_2 \right) - \frac{d_0}{2\,\mu \sqrt{\mu}} G_1 \right] \qquad (102)$$

which in turn reduces directly to

$$\frac{\partial \alpha_6(t)}{\partial \alpha_j(0)} = \frac{r_0}{r} f \frac{\partial r_0}{\partial \alpha_j(0)} + \frac{1}{r} g \frac{\partial d_0}{\partial \alpha_j(0)}$$

$$+ \frac{1}{2r} \left[- \mu g^2 - r\, G_2^2 + \frac{d}{\sqrt{\mu}} \left(\beta\, G_4 - 3\, G_5 + G_2\, G_3 \right) \right] \frac{\partial \frac{1}{a}}{\partial \alpha_j(0)}$$

$$+ \frac{1}{2 \mu r} \left[d\, \Delta t - d_0\, g \right] \frac{\partial \mu(0)}{\partial \alpha_j(0)} \quad . \qquad (103)$$

By expansion and collecting the right terms, it can be shown that the coefficient of $\frac{\partial \frac{1}{a}}{\partial \alpha_j(0)}$ in equation (103), which is in fact the $\Omega_{6,5}$ term, is exactly equivalent to the previously determined value of $\Omega_{6,5}$, namely

$$\Omega_{6,5} = \frac{1}{2r} \Big[- \mu g^2 - 3 \sqrt{\mu}\, g\, G_3 + r_0\, G_2^2 + \frac{d_0}{\sqrt{\mu}} \left(\beta G_4 - 3 G_5 + G_2 G_3 \right) + \left(\beta^2 G_4 - 3 \beta G_5 + 2 G_2 G_4 - 3 G_3^2 \right) \Big] \quad . \tag{104}$$

Finally, all that needs to be determined in order to have the complete Ω matrix is the four right-hand terms of the third row, namely $\frac{\partial \alpha_3(t)}{\partial \alpha_j(0)}$ for $j = 4, 5, 6$ and μ . Equation (81) can be written

$$\frac{\partial \alpha_3(t)}{\partial \alpha_j(0)} = \frac{1}{h v^2} \; H X \dot{R} \cdot \left\{ \dot{f} \frac{\partial R_0}{\partial \alpha_j(0)} + \dot{g} \frac{\partial \dot{R}_0}{\partial \alpha_j(0)} \right\} + \frac{h}{v^2} \left(\dot{f} \frac{\partial \dot{g}}{\partial \alpha_j(0)} - \dot{g} \frac{\partial \dot{f}}{\partial \alpha_j(0)} \right) \quad . \tag{105}$$

The first part of equation (105) uses the 4^{th}, 5^{th}, 6^{th} and 7^{th} columns of the $S(t_0)$ matrix respectively, depending on which $\alpha_j(0)$ we are using. So all that remains is to obtain an expression for the partials of $\dot{f}$ and $\dot{g}$. Utilizing the second form of $\dot{g}$ in equation (59) yields

$$\dot{f} \frac{\partial \dot{g}}{\partial \alpha_j(0)} - \dot{g} \frac{\partial \dot{f}}{\partial \alpha_j(0)} = \frac{\sqrt{\mu}}{r^2} \left(G_0 \frac{\partial G_1}{\partial \alpha_j(0)} - G_1 \frac{\partial G_0}{\partial \alpha_j(0)} \right) + \frac{\dot{f}\,\dot{g}}{r_0} \frac{\partial r_0}{\partial \alpha_j(0)}$$

$$+ \frac{\dot{f}\, G_0}{r} \frac{\partial r_0}{\partial \alpha_j(0)} - \frac{G_1^2}{r^2 r_0} \frac{\partial d_0}{\partial \alpha_j(0)}$$

$$- \frac{\dot{f}}{2 \mu r} \left(r - G_2 + \frac{d_0}{\sqrt{\mu}} G_1 \right) \frac{\partial \mu_0}{\partial \alpha_j(0)} \tag{106}$$

Equation (87) gives

$$G_0 \frac{\partial G_1}{\partial \alpha_j(0)} - G_1 \frac{\partial G_0}{\partial \alpha_j(0)} = \frac{\partial \beta}{\partial \alpha_j(0)} + \frac{1}{2} (G_3 + G_1 G_2) \frac{\partial \frac{1}{a}}{\partial \alpha_j(0)} \tag{107}$$

Making use of equations (107) and (90) reduces equation (106) to

$$
\dot{f}\,\frac{\partial \dot{g}}{\partial \alpha_j(0)} - \dot{g}\,\frac{\partial \dot{f}}{\partial \alpha_j(0)} = \frac{\dot{f}}{r}\left(\frac{r}{r_0}\,\dot{g} + G_0 + \frac{r_0}{r}\right)\frac{\partial r_0}{\partial \alpha_j(0)}
$$

$$
+\left(\frac{\dot{g}-1}{r^2} - \frac{r_0\dot{f}^2}{\mu}\right)\frac{\partial d_0}{\partial \alpha_j(0)}
$$

$$
+\frac{\sqrt{\mu}}{2r^3}\left[G_2\,(\sqrt{\mu}\,\Delta t + r\,G_1) + \beta G_4 - 3G_5\right]\frac{\partial \frac{1}{a}}{\partial \alpha_j(0)} \tag{108}
$$

$$
+\frac{1}{2\mu r}\left[\frac{\mu}{r^2}\left(\Delta t + \frac{d_0}{\mu}\,G_2\right) - \dot{f}\left(r - G_2 + \frac{d_0}{\sqrt{\mu}}\,G_1\right)\right]\frac{\partial \mu(0)}{\partial \alpha_j(0)}
$$

Using the $S(t_0)$ matrix (equation (53)), the general expression for the third row (equation (105)) and the respective parts of equation (108) we have

$$
\Omega_{3,4} = \frac{h}{v^2}\left[\frac{\dot{g}-1}{r^2} - \frac{r_0\dot{f}^2}{\mu} - \frac{\dot{f}}{h^2}\left(\frac{\mu}{r}\,g - d_0\right)\right] \tag{109}
$$

$$
\Omega_{3,5} = -\frac{\mu\dot{f}h}{2v^2v_0^2}\left[\dot{g} + \frac{d_0}{h^2}\left(\frac{\mu}{r}\,g - d_0\right)\right]
$$

$$
+\frac{\sqrt{\mu}\,h}{2r^3v^2}\left[G_2\,(\sqrt{\mu}\,\Delta t + r\,G_1) + \beta G_4 - 3G_5\right] \tag{110}
$$

$$
\Omega_{3,6} = \frac{h\dot{f}}{r_0v^2}\left[\frac{r_0}{r}\,G_0 + \frac{r_0^2}{r^2} - \frac{\mu\dot{g}}{r_0v_0^2} + \frac{\mu d_0}{h^2r_0v_0^2}\left(1 - \frac{r_0}{a}\right)\left(\frac{\mu}{r}\,g - d_0\right)\right] \tag{111a}
$$

$$\Omega_{3,7} = \frac{\dot{f} d_0}{2\mu h v^2}\left[\frac{\mu}{r} g - d_0 + \frac{h^2 r_0 \dot{f}}{\mu}\right] + \frac{h}{2 v^2 r^3}\left(\Delta t + \frac{d_0 G_2}{\mu}\right) \qquad (111b)$$

Picking up the results from equations (77), (80), (83), (109), (110), (111a), (111b), (99), (100), and (103) gives the complete Ω matrix

$$\Omega = \begin{bmatrix} \frac{v}{v_0} f & -\frac{v}{r_0} g & 0 & 0 & 0 & 0 & 0 \\ -\frac{r}{v_0} \dot{f} & \frac{r}{r_0} \dot{g} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & \Omega_{3,4} & \Omega_{3,5} & \Omega_{3,6} & \Omega_{3,7} \\ 0 & 0 & 0 & \dot{g} & \Omega_{4,5} & r_0 \dot{f} & \Omega_{4,7} \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{r} g & \Omega_{6,5} & \frac{r_0}{r} f & \Omega_{6,7} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad (112)$$

where $\Omega_{3,4}$, $\Omega_{3,5}$, $\Omega_{3,6}$ and $\Omega_{3,7}$ are equations (109), (110), (111a) and (111b) respectively and

$$\Omega_{4,5} = -\frac{3}{2}\mu \Delta t - \frac{1}{2}\mu g(\dot{g} - 2) + \frac{\sqrt{\mu}}{2r}\left(\beta G_4 - 3G_5 + G_2 G_3\right) \tag{113}$$

$$\Omega_{6,5} = \frac{1}{2r}\left[-\mu g^2 - r G_2^2 + \frac{d}{\sqrt{\mu}}\left(\beta G_4 - 3G_5 + G_2 G_3\right]\right. \tag{114}$$

$$\Omega_{4,7} = \frac{1}{2\mu}\left[d - d_0 \dot{g} + \frac{\mu \Delta t}{r}\left(\frac{r v^2}{\mu} - 1\right)\right] \tag{115}$$

$$\Omega_{6,7} = \frac{1}{2\mu r}\left(d \Delta t - d_0 g\right) . \tag{116}$$

VII REFERENCES

1. Analytical Mechanics Associates, Inc., "Final Report for Minimum Variance Precision Tracking and Orbit Prediction Program," May 1963.

2. Boyce, W. M., "Analytical Derivatives for a General Conic over Extended Time-Arcs," NASA Manned Spacecraft Center Memorandum, September 27, 1963.

3. Pines, S., Wolf, H., Bailie A., and Mohan J., "Modifications of the Goddard Minimum Variance Program for the Processing of Real Data," Analytical Mechanics Associates, Inc. Report, October 16, 1964.

4. Stumpff, K., Himmelsmechanik, VEB Verlag, Berlin, 1959.

5. Herrick, S., Astrodynamics, D. Van Nostrand Co., Inc., Princeton, New Jersey, in press.

6. Pines, S., Payne, M., and Wolf, H., "Asymptotically Singular Differential Correction Matrices,", 30 April 1961.